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Remark on mass and uniqueness conditions for homogeneous covariant equations

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Abstract. New mass and uniqueness theorems, given by Cotăescu for first- and second-order homogeneous covariant equations, yield new theories. It is shown that the new theories all exhibit zero charge and energy density.

Recently Cotăescu (1976) has given new necessary and sufficient mass and uniqueness conditions on first- and second-order homogeneous covariant equations (Tung 1967). These conditions allow for a wider class of unique mass equations than those previously treated. It is the object of this paper to point out that the new equations obtained will, however, have zero charge and energy density. For first-order equations this has been known for some time, while the analogous proof for second-order equations is given below.

The equations have the momentum-space form:

$$q^{\gamma}E(p/q)\psi(p)=m^{\gamma}\psi(p) \tag{1}$$

where $q=(p^2)^{1/2}$, $\gamma=1,2$ is the order of the equation, and $E(p/q)$ is a matrix homogeneous in p/q. For such equations to be covariant and to describe a particle with unique total spin, the field ψ is assumed to transform according to some, in general reducible, representation of the Lorentz group, and in the rest frame the matrix E becomes a projection operator onto eigenstates with this total spin. Thus in the rest frame, $p^{(0)}=(q,0,0,0)$, the components of a field with given spin j_0 and spin projection σ will be:

$$\psi_{\tau js}(p^{(0)},j_0,\sigma)=\xi^{\tau}\delta_{jj_0}\delta_{s\sigma} \tag{2}$$

where τ labels the representation $\mathscr{D}(k,l)$ of $\mathscr{L}_p$, j and s label the usual rotation basis eigenvectors, where $j=|k-l|,\ldots,|k+l|$ and $s=-j,\ldots,j$. In this basis E will be represented by

$$E_{\tau js,\tau'j's'}(p_0/q)=C^{\tau\tau'}\delta_{jj_0}\delta_{j'j_0}\delta_{ss'} \tag{3}$$

in the rest frame, where $C^{\tau\tau'}$ are constants and depend only on the representations τ, τ'. From (2) and (3), (1) becomes

$$\sum_{\tau'}C^{\tau\tau'}\xi^{\tau'}=\mu^{\gamma}\xi^{\tau} \tag{4}$$

in the rest frame, where $\mu=m/q$. This is the eigenvector problem for the $N\times N$ matrix

C where N is the number of representations of $\mathscr{L}_p$ used for ψ, and it generalizes the familiar Gel'fand–Yaglom treatment of first-order field equations (Gel'fand *et al* 1963).

For (1) to describe a single particle, or particle–antiparticle pair if $\gamma = 1$, we require a mass condition so that (1) only has solutions for $q^2 = m^2$, and further we require that the solution corresponding to this given mass should be unique. Thus, the mass-spin state should be non-degenerate and unique. First-order theories of this type have been studied by Cox (1974a, b, c). Cotăescu shows that these mass and uniqueness theorems can be expressed as the following condition on the matrix C of (3):

$$C^{r/\gamma}(C^{2/\gamma} - I)^k = 0 \qquad \text{(minimal equation)} \tag{5a}$$

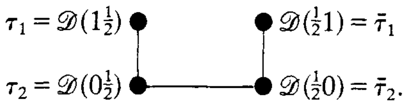
$$\mathrm{Rank}[C - (\pm 1)^\gamma I] = N - 1 \tag{5b}$$

where $r + 2k \leqslant \gamma N$ and r/γ is an integer. The case $k = 1$ corresponds to the usual Harish–Chandra unique-mass condition, and also that imposed by Tung. Equation (5*a*) ensures that only states with mass m can occur, while equation (5*b*) ensures that the state corresponding to this mass is unique.

As an example, Cotăescu considers the first-order equation

$$\beta_\mu p_\mu \psi(p) = m\psi(p) \tag{6}$$

corresponding to $E(p/q) = q^{-1}\beta_\mu p_\mu$, based on the representation:



This is the same representation as used for the Rarita–Schwinger spin-$\frac{3}{2}$ theory, but the absence of the linkage τ_1–$\bar{\tau}_1$ eliminates the spin-$\frac{3}{2}$ state, and the β_μ are chosen such that a unique spin-$\frac{1}{2}$ state is exhibited. The theory is completely determined by covariance and the form of β_0, and a number of theories can be obtained by fixing the minimal equation of β_0 in accordance with the conditions (5). The choice of $\beta_0(\beta_0^2 - I) = 0$ for minimal equation yields the spin-$\frac{1}{2}$ equations already studied by Capri (1969), while the choice

$$(\beta_0^2 - I)^2 = 0 \tag{7a}$$

$$\mathrm{Rank}(\beta_0 \pm I) = 3 \tag{7b}$$

yields new equations, for which β_0 has the form:

$$\beta_0 = \begin{array}{c} \begin{array}{cccc} \tau_1 & \tau_2 & \bar{\tau}_2 & \bar{\tau}_1 \end{array} \\ \begin{bmatrix} 0 & a & 0 & 0 \\ -1/a & 0 & 2 & 0 \\ 0 & 2 & 0 & -1/a \\ 0 & 0 & a & 0 \end{bmatrix} \end{array}. \tag{8}$$

We should point out, however, that this form, given by Cotăescu, fails to take account of the Lagrangian origin for the theory. In order that (6) is deducible from a real invariant

Lagrangian density it is necessary that an invariant Hermitian operator η should exist such that

$$\beta_0^\dagger \eta = \eta \beta_0 \tag{9}$$

(Tung 1967). In the representation (8) it is always possible to choose η to take the form

$$\eta = \begin{bmatrix} 0 & 0 & 0 & c \\ 0 & 0 & -c & 0 \\ 0 & -c & 0 & 0 \\ c & 0 & 0 & 0 \end{bmatrix} \tag{10}$$

where $c = \pm 1$ (η can only connect conjugate representations, and one can always scale the basis vectors to make the non-zero elements ± 1). Equation (9) can then be satisfied only if $a^2 = 1$, so this is an extra condition on β_0.

The theory defined by (8), with $a^2 = 1$, is thus derivable from a Lagrangian and satisfies the mass and uniqueness condition (5). However, this new spin-$\frac{1}{2}$ theory will have zero charge and energy density, because of the repeated factors corresponding to non-zero eigenvalues in the minimal equation. It is a well known result in the theory of first-order equations of the form (6) that the charge and energy density of the physical states of the field ψ will be non-zero if and only if the minimal polynomial of β_0 contains no repeated factors corresponding to non-zero eigenvalues (see for example Cox 1974a, b, c, Udgaonkor 1952, Speer 1969). Thus, the non-zero eigenvalues part of β_0 must be diagonalizable. Any non-diagonalizability of the β_0 (and this is essential for high-spin theories quantizable without use of an indefinite metric) must arise from repeated zero eigenvalues, which in fact correspond to the constraints in the theory.

For first-order theories the minimal polynomial of β_0 has to be of the form

$$m(\lambda) = \lambda^q \prod_{i=1}^{l} (\lambda^2 - m_i^2)$$

where $q + 2l \leqslant N$, and the m_i are distinct and non-zero. The uniqueness condition then implies that the characteristic polynomial of β_0 must be of the form

$$\Delta(\lambda) = \lambda^{N-2l} \prod_{i=1}^{l} (\lambda^2 - m_i^2)$$

(Cox 1974a, b, c). Thus, for first-order theories ($\gamma = 1$), all of Cotăescu's new equations arising from (5) with $k > 1$ will have zero charge and energy density—including his spin-$\frac{1}{2}$ example. We now show that this also holds for the second-order equation ($\gamma = 2$).

The form of (1) when $\gamma = 2$, in coordinate space, is

$$(\gamma^{\mu\nu} \partial_\mu \partial_\nu - m^2) \psi(x) = 0 \tag{11}$$

where $\gamma^{\mu\nu} = \gamma^{\nu\mu}$ are as given by Tung (1967). In the rest frame this equation becomes

$$(\gamma^{00} \partial_0^2 - m^2) \psi = 0$$

and if we take a plane wave representation for ψ, $\psi = \xi \, e^{ipx}$ this becomes the eigenvector problem for the matrix γ^{00} (cf (4))

$$(\gamma^{00} - \mu^2) \xi = 0 \tag{12}$$

where $\mu = m/q$, and ξ carries the spin indices. As usual, we suppress the spin and spin projection indices, and regard γ^{00} as an $N \times N$ matrix, and ξ as an $N \times 1$ vector. Thus, comparing with (4), $C = \gamma^{00}$.

The equation (11) can be derived from the Lagrangian density

$$\mathscr{L} = \tfrac{1}{2}(\bar{\psi}\gamma^{\mu\nu}\partial_\mu\partial_\nu\psi + \partial_\mu\partial_\nu\bar{\psi}\gamma^{\mu\nu}\psi) - m^2\bar{\psi}\psi \tag{13}$$

where $\bar{\psi} = \psi^\dagger\eta$, and $\gamma^{\mu\nu\dagger}\eta = \eta\gamma^{\mu\nu}$ (Tung 1967). The classical free charge-current density vector calculated from this is

$$j_\mu = \mathrm{i}\epsilon(\bar{\psi}\gamma^{\mu\nu}\partial_\mu\psi - \partial_\mu\bar{\psi}\gamma^{\mu\nu}\psi)$$

so the charge density is

$$\rho = \mathrm{i}\epsilon(\bar{\psi}\gamma^{\mu 0}\partial_\mu\psi - \partial_\mu\bar{\psi}\gamma^{\mu 0}\psi)$$

in natural units. We can now evaluate this in the rest frame, since if it vanishes in the rest frame it vanishes in all frames. Taking the plane wave representation for the field ψ, $\psi = \xi\,\mathrm{e}^{\mathrm{i}px}$, ρ becomes a quantity like

$$\rho \sim k\bar{\xi}\gamma^{00}\xi = k\mu^2\bar{\xi}\xi$$

where ξ is the eigenvector of γ^{00} corresponding to the eigenvalue μ^2, from (12). The energy density too will reduce to the form $c\bar{\xi}\xi$.

Now assume, with Cotăescu, that the minimal equation of γ^{00} is of the form

$$(\gamma^{00})^l(\gamma^{00} - 1)^k = 0 \tag{14}$$

where $k \geqslant 1$, so that the only non-zero value of μ^2 is 1, giving unique mass. Then the eigenvector corresponding to the eigenvalue 1 is

$$\xi = (\gamma^{00})^l(\gamma^{00} - 1)^{k-1}\phi$$

where ϕ is an arbitrary vector. (Clearly $\gamma^{00}\xi = \xi$, also by definition of the minimal polynomial ξ is non-zero, and by (5*b*) it is unique.) We therefore have

$$\begin{aligned}\bar{\xi}\xi &= \phi^\dagger(\gamma^{00})^{l\dagger}(\psi^{00\dagger} - 1)^{k-1}\eta(\gamma^{00})^l(\gamma^{00} - 1)^{k-1}\phi \\ &= \phi^\dagger\eta(\gamma^{00})^{2l}(\gamma^{00} - 1)^{2(k-1)}\phi\end{aligned}$$

and using (13), $\bar{\xi}\xi = 0$ if $k > 1$ by the minimal equation (14). We can only obtain non-zero charge or energy density when $k = 1$, which is the same conclusion as for the case of first-order equations.

We can verify this directly for Cotăescu's example, using the form corresponding to (10), which is derivable from a Lagrangian. The eigenvectors of β_0 are $\xi_\pm = (a, \pm 1, 1, \pm a)$ and substituting from (10) we obtain $\bar{\xi}_+\xi_+ = 0 = \bar{\xi}_-\xi_-$ using $a^2 = 1$. Thus the spin-$\frac{1}{2}$ example has zero charge and energy density, as do all theories satisfying (5), with $k > 1$.

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